# SIMPLICIAL ISOMETRIC EMBEDDINGS OF PSEUDO-METRIC POLYHEDRA

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Abstract. We give two definitions for a pseudo-metric polyhedron. We use the first definition to show that every pseudo-metric polyhedron (with the maximal degree of any vertex bounded above) admits a simplicial isometric embedding into Minkowski space of an appropriate signature. We then use this result to show that our two definitions for a pseudo-metric polyhedron coincide. Finally we use the second definition to prove that every pseudo-metric polyhedron admits a piecewise linear isometric embedding into Minkowski space of a (surprisingly low) signature.

## 1. Introduction

In the early to mid 1950's John Nash ([8] and [9]) solved the isometric embedding problem for Riemannian manifolds. In 1970 Robert Greene [3] and M. L. Gromov and V. A. Rokhlin [4] independently proved that every manifold with an indefinite metric admits an isometric embedding into Minkowski space of an appropriate signature. Nash's results with respect to isometric immersions have been successfully extended to Euclidean polyhedra. Some of the authors associated with this are Zalgaller [10], Krat [6], and Akopyan [1]. In a forthcoming paper [7] the author will extend the results of Akopyan to include embeddings. In the present we extend the results of Greene, Gromov and Rokhlin to polyhedra where each simplex is endowed with an indefinite metric tensor.

Let $\mathcal{X}$ be a topological space and let $\mathcal{T}$ be a fixed triangulation[1] of $\mathcal{X}$. We call the tuple $(\mathcal{X}, \mathcal{T})$ a *polyhedron*. When $\mathcal{X}$ is compact (so in particular $\mathcal{T}$ is finite) we call $(\mathcal{X}, \mathcal{T})$ a compact polyhedron. Let $\mathcal{V}$ denote the collection of vertices of $\mathcal{T}$ and let $\mathcal{E}$ denote the collection of edges of $\mathcal{T}$. We define a (piecewise-flat) pseudo-metric on $(\mathcal{X}, \mathcal{T})$ to simply be a function $g : \mathcal{E} \to \mathbb{R}$. A *pseudo-metric polyhedron* is just a triple $(\mathcal{X}, \mathcal{T}, g)$ satisfying the above. A pseudo-metric polyhedron has *dimension* $n$ if the simplicial complex $\mathcal{T}$ has dimension $n$. When there is no risk of ambiguity, we will denote either $(\mathcal{X}, \mathcal{T}, g)$ or $(\mathcal{X}, \mathcal{T})$ simply by $\mathcal{X}$.

Of course a pseudo-metric as defined above will, in general, not lead to a metric on $\mathcal{X}$. One obvious problem is if $g$ assigns a negative value to an edge. But a (slightly) more subtle problem is if the edge lengths of a $k$-dimensional simplex do not satisfy the "$k$-dimensional triangle inequality".

We will see in section 6 that the collection of pseudo-metrics as defined above are in 1-1 correspondence to an assignment of a (not necessarily positive-definite, or even non-degenerate) symmetric bilinear form to each simplex of $\mathcal{T}$ which "agrees" on intersecting simplices. We will make all of this precise then.

The following three theorems are proved in this paper.

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[1] All triangulations in this paper are simplicial

**Theorem 1.1.** *Let $(\mathcal{X}, \mathcal{T}, g)$ be a compact n-dimensional pseudo-metric polyhedron with vertex set $\mathcal{V}$. Let $d = max\{deg(v) | v \in \mathcal{V}\}$ and let $q = max\{d,\ 2n+1\}$. Then there exists a simplicial isometric embedding of $\mathcal{X}$ into $\mathbb{R}^q_q$.*

**Theorem 1.2.** *Let $(\mathcal{X}, \mathcal{T}, g)$ be a compact n-dimensional pseudo-metric polyhedron with edge set $\mathcal{E}$. Then $\mathcal{X}$ admits a simplicial isometric embedding into $\mathbb{R}^p_q$ for some integers p and q which satisfy $p + q = 2n + 1 + |\mathcal{E}|$ and $p \geq 2n + 1$.*

**Theorem 1.3.** *Let $(\mathcal{X}, \mathcal{T}, g)$ be an n-dimensional pseudo-metric polyhedron with vertex set $\mathcal{V}$ and suppose that $d = max\{deg(v) | v \in \mathcal{V}\} < \infty$. Let $q = max\{d,\ 2n+1\}$. Then there exists a simplicial isometric embedding of $\mathcal{X}$ into $\mathbb{R}^p_p$ where $p = 2q(d^3 - d^2 + d + 1)$.*

Notice that that Theorems 1.1 and 1.2 are essentially the same but, due to the existence of the $|\mathcal{E}|$ term in Theorem 1.2, the dimension requirements in Theorem 1.1 will generally be much smaller than those of Theorem 1.2. The reason we include Theorem 1.2 is because the proof is somewhat constructive while the proof of Theorem 1.1 is completely existential. The method of proof for Theorem 1.2 is also interesting in its own right though. We will prove Theorem 1.1 in section 3 and Theorem 1.2 in section 4. The proof of Theorem 1.3 uses Theorem 1.1 and will be done in section 5.

The above results are somewhat surprising because the embeddings are simplicial. Due to the presence of $d$ in the dimensions, it certainly may be that the dimensions involved are not strict. If we allow our isometric embeddings to be piecewise linear instead of simplicial, then by applying the results of Zalgaller [10] and Krat [6] with our second definition of a pseudo-metric polyhedron[2] and using a trick due to Greene [3] we can improve on these dimensions greatly.

**Theorem 1.4.** *Let $(\mathcal{X}, \mathcal{T}, g)$ be an n-dimensional pseudo-metric polyhedron where the triangulation $\mathcal{T}$ is locally finite. Then $\mathcal{X}$ admits a piecewise linear isometric embedding into $\mathbb{R}^n_{2n+1}$.*

For a proof of Theorem 1.4 see section 6. In a forthcoming paper [7] the author will prove that every $n$-dimensional Euclidean polyhedron admits a piecewise linear isometric embedding into $\mathbb{E}^{3n}$. Then using some of the same tricks we can reduce the dimension of Theorem 1.4 to $\mathbb{R}^n_{2n}$. The author finds the low dimensionality necessary to be rather surprising.

**Acknowledgements.** The motivation for many of the ideas in this paper comes from Nash [9] and Greene [3]. The author is indebted to their work.

## 2. Preliminaries

### 2.1. Definitions and Facts.

**Definition 2.1.** A set of $k$ points in $\mathbb{R}^N$ (with $k \leq N + 1$) is said to be *affinely independent* if the entire set of points is not contained in any $(k-2)$-dimensional affine subspace of $\mathbb{R}^N$. A set of points $\mathscr{A}$ in $\mathbb{R}^N$ is said to be in *general position* if every subset of $\mathscr{A}$ containing $N+1$ or fewer points is affinely independent. Suppose $n$ and $N$ are integers with $n \leq N$. A set of points $\mathscr{B}$ in $\mathbb{R}^N$ is said to be in *$n$-general position* if every subset of $\mathscr{B}$ containing $n+1$ or fewer points is affinely independent.

**Definition 2.2.** Let $(\mathcal{X}, \mathcal{T})$ be a polyhedron. *Simp($\mathcal{X}, \mathbb{R}^N$)* denotes the collection of all simplicial maps from $\mathcal{X}$ into $\mathbb{R}^N$ (with respect to $\mathcal{T}$) and *Met($\mathcal{X}$)* denotes the collection of all metrics on $\mathcal{X}$ as defined in section 1.

[2] See section 6

Notice that if $(\mathcal{X}, \mathcal{T})$ is a compact polyhedron and if we fix an ordering on the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$ that we have a bijective correspondence between $\mathrm{Simp}(\mathcal{X}, \mathbb{R}^N) \cong \mathbb{R}^{N|\mathcal{V}|}$ and $\mathrm{Met}(\mathcal{X}) \cong \mathbb{R}^{|\mathcal{E}|}$. This allows us to consider both $\mathrm{Simp}(\mathcal{X}, \mathbb{R}^N)$ and $\mathrm{Met}(\mathcal{X})$ as topological vector spaces. And this remark does not change if we replace the Euclidean inner product on $\mathbb{R}^N$ with any Minkowski inner product (see section 2.2).

**Lemma 2.3.** *Let $(\mathcal{X}, \mathcal{T})$ be an $n$-dimensional polyhedron, let $f \in Simp(\mathcal{X}, \mathbb{R}^N)$, and let $\mathcal{V}$ be the vertex set of $\mathcal{X}$. Let $f(\mathcal{V})$ denote the collection of the images of the vertices of $\mathcal{T}$. If $f(\mathcal{V})$ is in $(2n+1)$-general position (so in particular we must have $N \geq 2n+1$) then $f$ is an embedding.*

For a proof of Lemma 2.3 see [5]

A nice little excercise involving general position and Lemma 2.3 is the following:

**Exercise 2.4.** Let $(\mathcal{X}, \mathcal{T}, g)$ be a compact 1-dimensional *Euclidean*[3] polyhedron and let $\mathcal{T}'$ denote the barycentric subdivision of $\mathcal{T}$. Then $(\mathcal{X}, \mathcal{T}', g)$ admits a simplicial isometric embedding into $\mathbb{R}^4$.

## 2.2. **Minkowski Space $\mathbb{R}^p_q$.**

**Definition 2.5.** *Minkowski space of signature* $(p, q)$, denoted by $\mathbb{R}^p_q$, is $\mathbb{R}^{p+q}$ endowed with the symmetric bilinear form of signature $(p, q)$. More specifically, if $\vec{v}, \vec{w} \in \mathbb{R}^p_q$ with $\vec{v} = (v_i)_{i=1}^{p+q}$ and $\vec{w} = (w_i)_{i=1}^{p+q}$ then

$$< \vec{v}, \vec{w} >_{\mathbb{R}^p_q} := < \vec{v}, \vec{w} > := \sum_{i=1}^{p} v_i w_i - \sum_{i=p+1}^{q} v_i w_i$$

**Remarks:**

(1) In later parts of the paper we will not be as concerned as above with respect to the first $p$ coordinates of $\mathbb{R}^p_q$ being the "positive" coordinates with respect to $<,>$, and in general we will write

$$< \vec{v}, \vec{w} > = \sum_{i=1}^{p+q} \sigma(i) v_i w_i$$

where $\sigma(i) = 1$ for $p$ (fixed) coordinates and $\sigma(i) = -1$ for the other $q$ coordinates.

(2) By $\mathbb{R}^p_q$ we mean specifically $\mathbb{R}^{p+q}$ endowed with the symmetric bilinear form of signature $(p, q)$, by $\mathbb{E}^N$ we mean $\mathbb{R}^N$ with the symmetric bilinear form of signature $(N, 0)$, and by using $\mathbb{R}^N$ we mean to include the possibility of *any* Minkowski inner product of signature $(p', q')$ such that $p' + q' = N$.

(3) Define the *signed square* function $s : \mathbb{R} \to \mathbb{R}$ by $s(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

If $g \in \mathrm{Met}(\mathcal{X})$ then define $g^2 \in \mathrm{Met}(\mathcal{X})$ by $g^2(e) := s(g(e))$ for any edge $e$ of $\mathcal{T}$. Then a simplicial isometric embedding of $(\mathcal{X}, \mathcal{T}, g)$ into $\mathbb{R}^p_q$ is an embedding $h \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^p_q)$ which satisfies that for any edge $e_{ij} \in \mathcal{E}$ between vertices $v_i$ and $v_j$:

$$< (h(v_i) - h(v_j)), (h(v_i) - h(v_j)) > = g^2(e_{ij})$$

[3]We define this in Section 5. But for a 1-dimensional polyhedron all we mean is that $g(e) > 0$ for any edge $e$.

2.3. **The Inverse Function Theorem.** In the literature there is often ambiguity between the Inverse Function Theorem, the Implicit Function Theorem, the Constant Rank Theorem and other similar theorems. When we refer to the Inverse Function Theorem we are refering to the following[4]:

**Theorem 2.6.** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^1$ function and let $x \in \mathbb{R}^n$ be such that the rank of the differential of $F$ evaluated at $x$ is $m$ (so in particular we must have $n \geq m$). Then there exist open sets $U$ of $x$ and $V$ of $F(x)$ such that $F$ maps $U$ onto $V$. That is, for all $z \in V$ there exists $y \in U$ such that $F(y) = z$.*

## 3. Proof of Theorem 1.1

For the remainder of sections 3 and 4 $(\mathcal{X}, \mathcal{T}, g)$ will denote a *compact* metric polyhedron. So $\mathcal{T}$ is assumed to be finite here.

### 3.1. The Map $\varphi$.

**Definition 3.1.** Define

$$\varphi : \mathrm{Simp}(\mathcal{X}, \mathbb{R}^N) \to \mathrm{Met}(\mathcal{X})$$

to be the square of the induced metric map. That is, if $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^N)$ and $e_{ij} \in \mathcal{E}$, define

$$\varphi(f)(e_{ij}) =< (f(v_i) - f(v_j)), (f(v_i) - f(v_j)) >= \sum_{k=1}^{N} \sigma(k)(f_k(v_i) - f_k(v_j))^2$$

where $e_{ij}$ is the edge between the vertices $v_i$ and $v_j$ and $(f_k)_{k=1}^N$ are the component functions of $f$.

**Remarks:**

(1) As we have defined it above, the domain of the map $\varphi$ technically depends on $N$. But we will abuse notation and not consider this. So, for example, we can talk about 2 metrics $\varphi(h_1)$ and $\varphi(h_2)$ where $h_1 \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{N_1})$ and $h_2 \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{N_2})$ and $N_1 \neq N_2$. If we are discussing the square of an induced metric $\varphi(h)$, it will always be clear into what dimensional space that $h$ is defined.

(2) The reason that we consider the square of the induced metric map instead of just the induced metric map is because the square of the induced metric map is, in some sense, "linear over addition in $\mathrm{Met}(\mathcal{X})$". To make this precise, let $\alpha \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{N_1})$ and let $\beta \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{N_2})$. Since $\mathrm{Met}(\mathcal{X})$ is a vector space we can consider $\varphi(\alpha) + \varphi(\beta)$. Then:

$$(\varphi(\alpha) + \varphi(\beta))(e_{ij}) = \varphi(\alpha)(e_{ij}) + \varphi(\beta)(e_{ij}) =$$
$$\sum_{k=1}^{N_1} \sigma(k)(\alpha_k(v_i) - \alpha_k(v_j))^2 + \sum_{k=1}^{N_2} \sigma(k)(\beta_k(v_i) - \beta_k(v_j))^2 = \varphi(\alpha \oplus \beta)(e_j^i)$$

where $\alpha \oplus \beta \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{N_1+N_2})$ is the concatenation of the maps $\alpha$ and $\beta$.

(3) One easy property of the map $\varphi$, but one which we will use later, is $\varphi(\lambda f) = \lambda^2 \varphi(f)$ for all $\lambda \in \mathbb{R}$ and for all $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^N)$. To see this, just note that

$$\varphi(\lambda f)(e_{ij}) = \sum_{k=1}^{N} \sigma(k)(\lambda f_k(v_i) - \lambda f_k(v_j))^2 = \lambda^2 \sum_{k=1}^{N} \sigma(k)(f_k(v_i) - f_k(v_j))^2 = \lambda^2 \varphi(f)(e_{ij})$$

.

[4] I have seen this referred to as the local form of submersions, which is my preferred terminology

3.2. **The Differential of $\varphi$.** The next Lemma is crucial in proving Theorem 1.1.

**Lemma 3.2.** *Let $d = max\{deg(v)|v \in \mathcal{V}\}$ and let $f \in Simp(\mathcal{X}, \mathbb{R}^N)$ with $N \geq d$. If the images of the vertices of $\mathcal{T}$ under $f$ are in $d$-general position then the differential of $\varphi$ at $f$ has rank $|\mathcal{E}|$.*

*Proof.* If we fix $N \in \mathbb{N}$, then the Jacobian Matrix of $\varphi$ will be an $|\mathcal{E}| \times N|\mathcal{V}|$ matrix. So as a first observation note that for $d\varphi$ to be surjective at any point we must have $N|\mathcal{V}| \geq |\mathcal{E}| \Rightarrow N \geq \frac{|\mathcal{E}|}{|\mathcal{V}|}$. Let $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^N)$ with component functions $(f_k)_{k=1}^N$ and let $e_{ij}$ be an edge of $\mathcal{T}$ connecting the vertices $v_i$ and $v_j$. Let $\varphi_{e_{ij}}$ denote the $e_{ij}$ component of the map $\varphi$ (thought of as a map from $\mathbb{R}^{N|\mathcal{V}|}$ to $\mathbb{R}^{|\mathcal{E}|}$) and let $f_k^i := f_k(v_i)$ $\forall\, 1 \leq i \leq |V|$ and $\forall\, 1 \leq k \leq N$. Note that in this notation $\varphi_{e_{ij}}(f) = \sum_{k=1}^N \sigma(k)(f_k^i - f_k^j)^2$. Then we compute:

$$\frac{\partial \varphi_{e_{ij}}}{\partial f_k^l} = \begin{cases} 0 & \text{if } l \neq i, j \\ 2\sigma(k)(f_k^i - f_k^j) & \text{if } l = i \\ 2\sigma(k)(f_k^j - f_k^i) & \text{if } l = j \end{cases}$$

To prove Lemma 3.2 we need for the rows of $d\varphi|_f$ to be linearly independent when considered as vectors in $\mathbb{R}^{N|\mathcal{V}|}$. Since multiplying a column of a matrix by a non-zero constant does not change the rank of the matrix, we can multiply each column of $d\varphi|_f$ by -1 if necessary in order to remove the $\sigma(k)$. For an arbitrary edge $e_{ij}$ of $\mathcal{E}$, we need to analyze the row of $d\varphi|_f$ corresponding to this edge.

The matrix $d\varphi|_f$ has $N|\mathcal{V}|$ columns. But it's easier to see what is happening if we consider $d\varphi|_f$ to have $|\mathcal{V}|$ columns, the entries of which are row vectors of $\mathbb{R}^N$. We will call these **block columns** of $d\varphi|_f$ (so in particular $d\varphi|_f$ has $|\mathcal{V}|$ block columns). Using this notation, we can see that the row of $d\varphi|_f$ corresponding to the edge $e_{ij}$ looks like:

$$\left[\vec{0}|.....|\vec{0}|2(f(v_i) - f(v_j))|\vec{0}|.....|\vec{0}|2(f(v_j) - f(v_i))|\vec{0}|.....|\vec{0}\right]$$

where the vertical lines are intended to break up the row into $|V|$ block columns. The $2(f(v_i)-f(v_j))$ occurs in the $i^{th}$ block column and similarly $2(f(v_j)-f(v_i))$ is in the $j^{th}$ block column. Notice that if $f(v_i) = f(v_j)$, then this is the 0 row and $d\varphi|_f$ is therefore not surjective. So another necessary condition for $d\varphi|_f$ to be surjective is that $f(v_i) \neq f(v_j)$ for all adjacent vertices $v_i, v_j \in \mathcal{V}$.

Since multiplying a matrix by a non-zero scalar doesn't change the rank, we can let $(d\varphi|_f)^* = \frac{-1}{2} d\varphi|_f$ and work with $(d\varphi|_f)^*$ instead. Then the row of $(d\varphi|_f)^*$ corresponding to the edge $e_{ij}$ is:

$$\left[\vec{0}|.....|\vec{0}|f(v_j) - f(v_i)|\vec{0}|.....|\vec{0}|f(v_i) - f(v_j)|\vec{0}|.....|\vec{0}\right]$$

Notice that the $i^{th}$ entry of the row corresponding to the edge $e_{ij}$ is $f(v_j) - f(v_i) := f(e_{ij})$. This is just the vector in $\mathbb{R}^N$ whose initial point is $f(v_i)$ and whose terminal point is $f(v_j)$.

Now to see whether or not $(d\varphi|_f)^*$ is surjective, consider the block column corresponding to the vertex $v_i$. The non-zero entries of this column correspond exactly to the edges of $f(\mathcal{X})$ (considered as vectors in $\mathbb{R}^N$ that point away from $f(v_i)$) that are incident with the vertex $f(v_i)$. So if the set of edges of $f(\mathcal{X})$ incident with $f(v_i)$ is linearly independent then the block column corresponding to the vertex $v_i$ will have maximal rank (when considered as an $|\mathcal{E}| \times N$ matrix). Then let $d = \max\{\deg(v)|v \in \mathcal{V}\}$ and suppose that $N \geq d$. (Note in particular that if $N \geq d$ then $N|\mathcal{V}| \geq |\mathcal{E}|$.) Then for the block column of $(d\varphi|_f)^*$ corresponding to the vertex $v_i$, the rank of the

block column will be $\leq \min\{\deg(v_i),\, N\} = \deg(v_i)$. So if the set of edges of $f(\mathcal{X})$ incident with $f(v_i)$ are linearly independent, then the rank of the block column of $(d\varphi|_f)^*$ corresponding to the vertex $v_i$ will equal $\deg(v_i)$. Or, in other words, the rows of $(d\varphi|_f)^*$ corresponding to edges of $\mathcal{X}$ which are incident with $v_i$ are linearly independent. So if the set of edges of $f(\mathcal{X})$ at *every* vertex is linearly independent then $(d\varphi|_f)^*$ will have rank equal to $|\mathcal{E}|$ and will therefore be surjective. This criteria is met if the images of the vertices of $\mathcal{T}$ under $f$ are in $d$-general position.

□

Lemma 3.2 motivates the following definition:

**Definition 3.3.** An embedding of $\mathcal{X}$ into $\mathbb{R}^N$ whose vertices are in $d$-general position is called a *free embedding.*

On a historical note, John Nash in [9] and M.L. Gromov with V.A. Rokhlin in [4] study embeddings of manifolds where a more general inverse function theorem applies. Nash called these embeddings *perturbable* because he was developing a generalization of the inverse function theorem in which he would perturb these embeddings to induce the metric change that he wanted. But later Gromov and Rokhlin called these embeddings *free* because that more closely described the property that the embedding had to satisfy. And that property in the case of manifolds was that the collection of first and second order partial derivatives of the embedding function be linearly independent at every point, which is very similar to the property that we need in the case of embeddings of polyhedra. So we use the same terminology as [4] to be consistent.

An easy observation is the following:

**Lemma 3.4.** *Let $(\mathcal{X}, \mathcal{T})$ be a compact $n$-dimensional polyhedron with vertex set $\mathcal{V}$ and let $d = max\{deg(v) | v \in \mathcal{V}\}$. Let $N \geq max\{d, 2n+1\}$ and endow $Simp(\mathcal{X}, \mathbb{R}^N)$ with the canonical Lebesgue measure from $\mathbb{R}^{N|\mathcal{V}|}$. Then the collection of maps which are* ***not*** *free embeddings has measure 0. Thus, $(d\varphi|_f)^*$ is surjective for almost all $f$ in $Simp(\mathcal{X}, \mathbb{R}^N)$.*

3.3. **Proof of Theorem 1.1.** We are now ready to prove Theorem 1.1. What follows is due to a trick by Greene in [3]

*Proof of Theorem 1.1.* Let $f$ be a free simplicial embedding of $\mathcal{X}$ into $\mathbb{R}^q$, the existence of which is guaranteed by Lemma 3.4. Then the map $f \oplus f : \mathcal{X} \to \mathbb{R}^q_q \cong \mathbb{R}^{q+q}$ induces the 0 metric in $\mathbb{R}^q_q$, that is $\varphi(f) = 0$. $f \oplus f$ is free since $f$ is. So by the Inverse Function Theorem there exists a neighborhood $U$ of $f \oplus f$ in $\mathrm{Simp}(\mathcal{X}, \mathbb{R}^q_q)$ and a neighborhood $V$ of $\vec{0}$ in $\mathrm{Met}(X)$ such that $\varphi$ maps $U$ onto $V$. Note that since $f \oplus f$ is an embedding, we can choose $U$ to be a open neighborhood in the set of embeddings of $\mathcal{X}$ into $\mathbb{R}^q_q$.

Now, choose $\lambda > 0$ large enough so that $\frac{g^2}{\lambda^2} \in V$. Then there exists an embedding $h \in U$ such that $\varphi(h) = \frac{g^2}{\lambda^2}$. So $\lambda^2 \varphi(h) = g^2$ and thus $\varphi(\lambda h) = g^2$. Therefore, $\lambda h$ is an isometric embedding of $\mathcal{X}$ into $\mathbb{R}^q_q$.

□

## 4. Proof of Theorem 1.2

### 4.1. **Simplicial Maps with Spanning Metrics.**

**Definition 4.1.** Let $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^N)$ and let $(f_k)_{k=1}^l$ be the component functions of $f$ where $\forall\ 1 \leq k \leq l$, $f_k \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{N_k})$. Note that $\varphi(f_k) \in \mathrm{Met}(\mathcal{X})\ \forall k$ and $N = \sum_{k=1}^l N_k$. We say that $f$ has a **spanning metric** if $\{\varphi(f_k)\}_{k=1}^l$ spans $\mathrm{Met}(\mathcal{X})$.

**Lemma 4.2.** *Let $m < |\mathcal{E}|$ and suppose that the set $\mathbb{A} = \{g_1, g_2, ..., g_m\} \subset Met(\mathcal{X})$ is linearly independent. Then the set $\mathbb{B} = \{h \in Simp(\mathcal{X}, \mathbb{R}^d) | \mathbb{A} \cup \{\varphi(h)\}$ is linearly independent$\}$ is dense in $Simp(\mathcal{X}, \mathbb{R}^d)$, where $d = max\{deg(v) | v \in \mathcal{V}\}$.*

*Proof.* Let $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^d)$ and let $\epsilon > 0$. We need to construct $h \in \mathbb{B}$ such that $|f - h| < \epsilon$, where $|f - h|$ denotes the Euclidean metric on $\mathrm{Simp}(\mathcal{X}, \mathbb{R}^d) \cong \mathbb{R}^{d|\mathcal{V}|}$. By Lemma 3.4, almost all $f' \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^d)$ are free. So choose $f' \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^d)$ free such that $|f - f'| < \frac{\epsilon}{2}$. Now consider $f'$. If $f' \in \mathbb{B}$ then we are done. So suppose that $f' \notin \mathbb{B}$, which in particular means that $\varphi(f') \in \mathrm{Span}(\mathbb{A})$. Since $f'$ is free, $\exists$ neighborhoods $U$ of $f'$ and $V$ of $\varphi(f')$ such that $\varphi$ maps $U$ onto $V$. By intersecting $U$ with the sphere of radius $\frac{\epsilon}{2}$ centered at $f'$, we may assume that $U$ is contained in the sphere of radius $\frac{\epsilon}{2}$ centered at $f'$. Then since $\mathrm{Span}(\mathbb{A})$ is contained in a $|\mathcal{E}| - 1$ dimensional subspace of $\mathrm{Met}(\mathcal{X})$, it has measure 0 in $\mathrm{Met}(\mathcal{X})$ and therefore almost all points of $V$ do not lie in $\mathrm{Span}(\mathbb{A})$. So choose $\alpha \in V \setminus \mathrm{Span}(\mathbb{A})$. Then by the Inverse Function Theorem $\exists h \in U$ such that $\varphi(h) = \alpha$. So $h \in \mathbb{B}$ and $|f - h| \leq |f - f'| + |f' - h| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

□

**Corollary 4.3.** *There exists a simplicial map with a spanning metric in $Simp(\mathcal{X}, \mathbb{R}^{|\mathcal{E}|})$.*

**Remark:** Notice that if $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{|\mathcal{E}|})$ with component functions $\{f_k\}_{k=1}^{|\mathcal{E}|}$ has a spanning metric then the collection $(\varphi(f_k))_{k=1}^{|\mathcal{E}|}$ is a basis for $\mathrm{Met}(\mathcal{X})$.

*Proof.* We construct the component functions of the simplicial map with a spanning metric $f = \{f_k\}_{k=1}^{|\mathcal{E}|}$ recursively. Define $f_1 : \mathcal{X} \to \mathbb{R}$ to be any simplicial map which does not map all of the vertices of $\mathcal{T}$ to the same point (and thus does not induce the 0 metric). So $\varphi(f_1) \neq \vec{0}$ in $\mathrm{Met}(\mathcal{X})$.

Now suppose $f_1, ..., f_i$ have been defined for some $i < |\mathcal{E}|$ in such a way that the collection $\{\varphi(f_k)\}_{k=1}^{i}$ is linearly independent. Thus the collection $\{\varphi(f_k)\}_{k=1}^{i}$ does not span $\mathrm{Met}(\mathcal{X})$, so by Lemma 4.2 there exists $g \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^d)$ such that $\{\varphi(f_k)\}_{k=1}^{i} \cup \{\varphi(g)\}$ is also linearly independent. Let $\{g_l\}_{l=1}^{d}$ denote the component functions of $g$. We know that $\varphi(g) = \sum_{l=1}^{d} \varphi(g_l)$ so there must exist some component function $g_j$ such that $\varphi(g_j)$ is not in $\mathrm{Span}(\{\varphi(f_k)\}_{k=1}^{i})$. Choose $f_{i+1} = g_j$. Then the collection $\{\varphi(f_k)\}_{k=1}^{i+1}$ is linearly independent.

In this way we construct a function $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{|\mathcal{E}|})$ so that the collection of component functions under $\varphi$, $\{\varphi(f_k)\}_{k=1}^{|\mathcal{E}|}$, is linearly independent and thus spans $\mathrm{Met}(\mathcal{X})$. Therefore $f$ is a simplicial map with a spanning metric.

□

### 4.2. **Proof of Theorem 1.2.**

*Proof of Theorem 1.2.* Let $f \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{2n+1})$ be an embedding and let $h \in \mathrm{Simp}(\mathcal{X}, \mathbb{R}^{|\mathcal{E}|})$ be a simplicial map with a spanning metric, whose existence is guaranteed by Corollary 4.3. Let $\{h_k\}_{k=1}^{|\mathcal{E}|}$ be the component functions of $h$. Then by assumption, $\{\varphi(h_k)\}_{k=1}^{|\mathcal{E}|}$ spans $\mathrm{Met}(\mathcal{X})$. So $\exists \alpha_1, \alpha_2, ..., \alpha_{|\mathcal{E}|} \in \mathbb{R}$ such that

$$g^2 - \varphi(f) = \sum_{k=1}^{|\mathcal{E}|} \alpha_k \varphi(h_k).$$

Thus

$$g^2 = \varphi(f) + \sum_{k=1}^{|\mathcal{E}|} \alpha_k \varphi(h_k).$$

Let $p = 2n+1+|\{\alpha_k|\alpha_k \geq 0\}|$ and let $q = |\{\alpha_k|\alpha_k < 0\}|$. Then define $z \in \text{Simp}(\mathcal{X}, \mathbb{R}^p_q)$ by

$$z = f \bigoplus_{k=1\,,\,\alpha_k \geq 0}^{|\mathcal{E}|} \sqrt{\alpha_k} h_k \bigoplus_{l=1\,,\,\alpha_l<0}^{|\mathcal{E}|} \sqrt{|\alpha_l|} h_l$$

and notice that

$$\varphi(z) = \varphi(f) + \sum_{k=1\,,\,\alpha_k\geq 0}^{|\mathcal{E}|} \alpha_k \varphi(h_k) - \sum_{l=1\,,\,\alpha_l<0}^{|\mathcal{E}|} |\alpha_l| \varphi(h_l) = \varphi(f) + \sum_{k=1\,,\,\alpha_k\geq 0}^{|\mathcal{E}|} \alpha_k \varphi(h_k) + \sum_{l=1\,,\,\alpha_l<0}^{|\mathcal{E}|} \alpha_l \varphi(h_l)$$

$$= \varphi(f) + \sum_{k=1}^{|\mathcal{E}|} \alpha_k \varphi(h_k) = g^2$$

Therefore, $z$ is a simplicial isometry of $\mathcal{X}$ into $\mathbb{R}^p_q$ where $p+q = 2n+1+|\mathcal{E}|$ and $p \geq 2n+1$. $z$ is an embedding since $f$ is.

□

## 5. Proof of Theorem 1.3

For this section let $(\mathcal{X}, \mathcal{T}, g)$ be an $n$-dimensional pseudo-metric polyhedron with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, assume that $d = \max\{\deg(v)|v \in \mathcal{V}\} < \infty$, and let $q = \max\{2n+1, d\}$. For a vertex $v$ the closed star of $v$ will be denoted by $St(v)$. We define $St^2(v) := \bigcup_{u \in St(v)} St(u)$ and for any $k \in \mathbb{N}$ we define $St^{k+1}(v) := \bigcup_{u\in St^k(v)} St(u)$.

An outline of the proof is as follows. We first construct, for each $v \in \mathcal{V}$, a compact pseudo-metric polyhedron denoted by $(\mathcal{S}_v, \mathcal{T}_v, \gamma_v)$. We then partition $\mathcal{V}$ into $d^3 - d^2 + d + 1$ classes $\{\mathcal{C}_i\}_{i=1}^{d^3-d^2+d+1}$ which satisfy that if $u, v \in \mathcal{C}_i$ then[5] $u \notin St^3(v)$. Note that this is equivalent to the statement $int(St^2(u)) \cap int(St^2(v)) = \emptyset$ where by $int()$ we mean "interior". Now, for each $v$ in a fixed class $\mathcal{C}_i$ we construct a simplicial isometric embedding $\alpha_v : \mathcal{S}_v \to \mathbb{R}^{2q}_{2q}$ which satisfies that if $u, v \in \mathcal{C}_i$ then $\alpha_u(\mathcal{S}_u) \cap \alpha_v(\mathcal{S}_v) = \vec{0}$. This will allow us to define, for each $1 \leq i \leq d^3 - d^2 + d + 1$, a simplicial map $\beta_i : \mathcal{X} \to \mathbb{R}^{2q}_{2q}$. Then our simplicial isometric embedding will be

$$\lambda := \bigoplus_{i=1}^{d^3-d^2+d+1} \beta_i : \mathcal{X} \longrightarrow \mathbb{R}^p_p$$

where $p = 2q(d^3 - d^2 + d + 1)$.

5.1. **Construction of the Compact Pseudo-Metric Polyhedron $(\mathcal{S}_v, \mathcal{T}_v, \gamma_v)$.** Let $v \in \mathcal{V}$. The Polyhedron $\mathcal{S}_v$ will look like the cone of $St(v)$, and in fact that would work for our purposes. But in an attempt to keep the dimension of the embedding space as small as possible we alter the construction some as follows.

We describe $\mathcal{S}_v$ and the triangulation $\mathcal{T}_v$ at the same time. We start the construction of $\mathcal{S}_v$ with the entire complex $St(v)$. We then adjoin a vertex denoted by $v^*$ as follows. We glue in an edge between $v^*$ and a vertex $u$ on the boundary of $St(v)$ if and only if there exists an edge in $\mathcal{T}$ which is adjacent to $u$ and *not* contained in $St(v)$. We do *not* connect an edge between $v$ and $v^*$. Then for any $2 \leq k \leq n$ if there exist $k$ vertices on the boundary of $St(v)$ which are contained in the boundary of a $k$ simplex in $\mathcal{X} \setminus int(St(v))$ we glue in a $k$-dimensional simplex using those $k$ vertices and $v^*$ (see Figure 1).

[5]One easily sees that this condition is symmetric

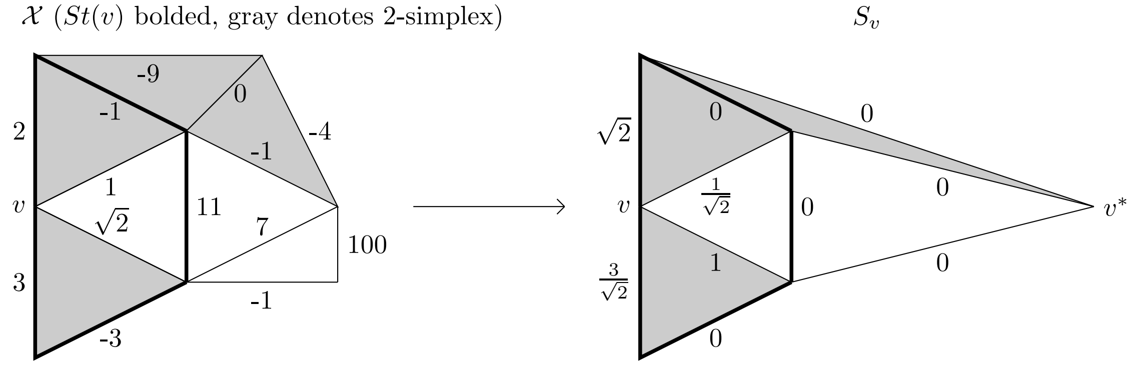


FIGURE 1. The assignments of $g$ and $\gamma_v$, respectively, are denoted along each edge.

This completes the construction of the polyhedron $(\mathcal{S}_v, \mathcal{T}_v)$. It is clear that $\mathcal{S}_v$ is compact. It is important to note that $\mathcal{S}_v$ has dimension less than or equal to $n$ and the maximal degree of any vertex is less than or equal to $d$, so it meets the criteria of Theorem 1.1. What is left to do is to describe the pseudo-metric $\gamma_v$.

Let $e$ be an edge of $\mathcal{S}_v$. Either $e$ is adjacent to $v$ or $e$ is not adjacent to $v$. In the latter case we simply define $\gamma_v(e) := 0$. So, in particular, notice that every edge adjacent to $v^*$ has intrinsic length 0. In the former case, when we consider $St(v)$ as a subcomplex of $\mathcal{X}$, our edge $e$ has an intrinsic length $g(e)$. We then define $\gamma_v(e) := \frac{1}{\sqrt{2}}g(e)$ (see Figure 1).

5.2. **Partitioning $\mathcal{V}$ into $(d^3 - d^2 + d + 1)$ Classes and Isometric Embeddings of $\mathcal{S}_v$ into $\mathbb{R}^{2q}_{2q}$.** Our first goal here is to partition $\mathcal{V}$ into $d^3 - d^2 + d + 1$ classes $\{\mathcal{C}_i\}_{i=1}^{d^3-d^2+d+1}$ which satisfy that if $u, v \in \mathcal{C}_i$ then $u \notin St^3(v)$. Since $\mathcal{T}$ is locally finite, $\mathcal{V}$ is countable. So enumerate $\mathcal{V}$ in some way. We determine which class a given vertex is in recursively. Put the first vertex in $\mathcal{C}_1$. Then assume that the classes of all of the vertices before a given vertex $v$ have been determined. This vertex $v$ is connected by an edge to at most $d$ other vertices of $\mathcal{T}$. Each of these vertices is connected by an edge to at most $d-1$ vertices of $\mathcal{T}$ other than $v$ (the $d-1$ is since they are all connected to $v$). Thus $\partial St^2(v)$ contains at most $d(d-1)$ vertices and $St^2(v)$ contains at most $d(d-1)+d = d^2$ vertices other than $v$. But each vertex in the boundary of $St^2(v)$ is connected by an edge to at most $d-1$ vertices not in $St^2(v)$ (by the same logic as before). So $St^3(v)$ contains at most $d(d-1)^2 + d^2 = d^3 - d^2 + d$ vertices other than $v$. Then since there are $d^3 - d^2 + d + 1$ classes, we can always find a class $\mathcal{C}_i$ such that $St^3(v)$ does not contain any of the vertices already in $\mathcal{C}_i$. So we place $v$ into such a class, completing the definition of the classes $\{\mathcal{C}_i\}_{i=1}^{d^3-d^2+d+1}$.

For the following discussion let us fix a class $\mathcal{C}_i$. By Theorem 1.1, for each $v \in \mathcal{C}_i$ there exists a simplicial isometric embedding $h_v : \mathcal{S}_v \to \mathbb{R}^q_q$. By composing with a translation we may assume that $h_v(v^*) = \vec{0}$. In what follows we construct, for each $v \in \mathcal{C}_i$, a (linear) isometric embedding $\iota_v : \mathbb{R}^q_q \to \mathbb{R}^{2q}_{2q}$ (see Figure 2). Then $\alpha_v := \iota_v \circ h_v$ will be our simplicial isometric embedding of $\mathcal{S}_v$ into $\mathbb{R}^{2q}_{2q}$.

Since $\mathcal{V}$ is countable, $\mathcal{C}_i$ is countable. So there exists an injection $\mu : \mathcal{C}_i \to \mathbb{N}$. Thinking of $\mathbb{R}^{2q}_{2q}$ as $\mathbb{R}^q_q \times \mathbb{R}^q_q$, define $\iota_v : \mathbb{R}^q_q \to \mathbb{R}^{2q}_{2q}$ by

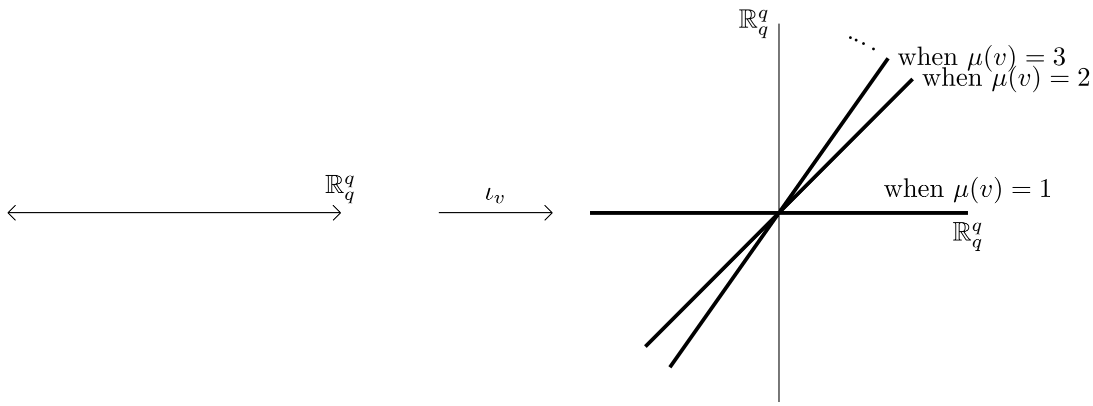


FIGURE 2. The images of $\iota_v$ for various values of $\mu(v)$.

$$\iota_v(\vec{x}) = \left( \sqrt{\frac{1}{\mu(v)}}\, \vec{x}, \sqrt{1 - \frac{1}{\mu(v)}}\, \vec{x} \right)$$

$\iota_v$ is a linear isometry, as desired. Note that since $h_v(v^*) = \vec{0}$, $\alpha_v(v^*) = \iota_v(\vec{0}) = \vec{0}$. Also notice that for $v, w \in \mathcal{C}_i$ we have that $\alpha_v(\mathcal{S}_v) \cap \alpha_w(S_w) = \vec{0}$

5.3. **Wrapping up the Construction.** What is left of the proof is to use the $\alpha_v' s$ of a class $\mathcal{C}_i$ to construct the simplicial map $\beta_i : \mathcal{X} \to \mathbb{R}^{2q}_{2q}$, and then showing that $\bigoplus_{i=1}^{d^3-d^2+d+1} \beta_i : \mathcal{X} \to \mathbb{R}^p_p$ where $p = 2q(d^3 - d^2 + d + 1)$ is an isometric embedding.

Since $\beta_i$ is to be simplicial, we need only define it on the vertices of $\mathcal{T}$. So let $u \in \mathcal{V}$ be arbitrary. If $u \in St(v)$ for some $v \in \mathcal{C}_i$ then define $\beta_i(u) := \alpha_v(u)$ (where, for $\alpha_v(u)$ to make sense, we are considering $St(v)$ as a subcomplex of $\mathcal{S}_v$). Otherwise, $u \notin St(v)$ for all vertices $v \in \mathcal{C}_i$. In this case we define $\beta_i(u) := \vec{0}$. This is a well-defined construction since the closed stars of vertices in $\mathcal{C}_i$ are disjoint.

**Showing $\lambda$ is an Isometry.** Now, for brevity, let $\lambda := \bigoplus_{i=1}^{d^3-d^2+d+1} \beta_i$. In order to show that $\lambda$ is an isometry, we need to analyze[6] $\varphi(\beta_i)(e)$ for each $i$ on each edge $e \in \mathcal{E}$.

So let $e \in \mathcal{E}$ be arbitrary, and let $u$ and $v$ denote the vertices adjacent to $e$, respectively. We break this down into four cases. The first and most important case is when one of the vertices $u$ or $v$ is in $\mathcal{C}_i$. Without loss of generality assume that it is $v$. Then $u, v \in St(v)$ and thus $\beta_i(v) = \alpha_v(v)$ and $\beta_i(u) = \alpha_v(u)$. So $\varphi(\beta_i)(e) = \varphi(\alpha_v)(e) = s(\gamma_v(e)) = \frac{1}{2}s(g(e)) = \frac{1}{2}g^2(e)$.

For the last three cases we assume that neither $u$ nor $v$ is in $\mathcal{C}_i$. Case 2 is when there exists $w \in \mathcal{C}_i$ such that both $u, v \in St(w)$, or equivalently $e \subseteq \partial St(w)$. This case is analogous to above but this time, due to the definition of $\gamma_w$, we have that $\varphi(\beta_i)(e) = s(\gamma_w(e)) = 0$. For Case 3 we assume that there exists $w \in \mathcal{C}_i$ such that exactly one of $u$ or $v$ is in $St(w)$, say $u \in St(w)$. It is important to note that there cannot exist $x \in \mathcal{C}_i$ such that $v \in St(x)$, for otherwise we would have $x \in St^3(w)$ which violates how we constructed the class $\mathcal{C}_i$. So here we see that $\beta_i(u) = \alpha_w(u)$ and

[6]for the definition of $\varphi$, please see section 3

$\beta_i(v) = \vec{0} = \alpha_w(w^*)$. Therefore $\varphi(\beta_i)(e) = \varphi(\alpha_w)(e)$[7]$= 0$. The last case is when neither $u$ nor $v$ is in the closed star of any member of $\mathcal{C}_i$. But in this case both vertices are mapped to $\vec{0}$ and hence $\varphi(\beta_i)(e) = 0$.

The key point to note here is that the only edges $e \in \mathcal{E}$ for which $\varphi(\beta_i)(e) \neq 0$ are those which are adjacent to a member of $\mathcal{C}_i$. And in this case $\varphi(\beta_i)(e) = \frac{1}{2}g^2(e)$. But since each edge is adjacent to exactly two vertices (both of which are in different classes), and since $\varphi$ is additive with respect to concatenation of maps, we see that for every edge $e \in \mathcal{E}$ we have that $\varphi(\lambda)(e) = \sum_{i=1}^{d^3-d^2+d+1} \varphi(\beta_i)(e) = \frac{1}{2}g^2(e) + \frac{1}{2}g^2(e) = g^2(e)$. Hence $\lambda$ is an isometry.

**Showing $\lambda$ is an Embedding.** Let $x, y \in \mathcal{X}$ with $x \neq y$. Let $v \in \mathcal{V}$ be such that $x \in int(St(v))$ and let $i$ be the index such that $v \in \mathcal{C}_i$. Note that since $x$ is in the interior of $St(v)$, $\beta_i(x) = \alpha_v(x) \neq \vec{0}$. What we will show is that $\beta_i(x) \neq \beta_i(y)$ and therefore $\lambda(x) \neq \lambda(y)$.

Clearly if $y \in St(v)$ then $\alpha_v(x) \neq \alpha_v(y) \implies \beta_i(x) \neq \beta_i(y) \implies \lambda(x) \neq \lambda(y)$. So suppose $y \notin St(v)$. If $y \in St^2(w)$ for any $w \in \mathcal{C}_i$ with $w \neq v$ then $\beta_i(y) = \alpha_w(y)$. But $\alpha_v(\mathcal{S}_v) \cap \alpha_w(\mathcal{S}_w) = \vec{0}$ and $\beta_i(x) \neq \vec{0}$, thus $\beta_i(x) \neq \beta_i(y)$. If $y \notin St^2(w)$ for any $w \in \mathcal{C}_i$ (including $v$) then $\beta_i(y) = \vec{0}$ and thus $\beta_i(y) \neq \beta_i(x)$.

So the only case left is when $y \in St^2(v) \setminus St(v)$. We can define a simplicial map[8] $\pi_v : \mathcal{X} \to \mathcal{S}_v$ by mapping each vertex of $St(v)$ to itself and mapping every other vertex of $\mathcal{T}$ to $v^*$. Note that $\pi_v$ maps all of $\mathcal{X} \setminus St^2(v)$ to $v^*$. So for all $v \in \mathcal{V}$ we have the following sequence of maps:

$$\mathcal{X} \xrightarrow{\ \pi_v\ } \mathcal{S}_v \xrightarrow{\ h_v\ } \mathbb{R}^q_q \xrightarrow{\ \iota_v\ } \mathbb{R}^{2q}_{2q}$$

When restricted to $St^2(v)$ it is easy to see that $\beta_i = \iota_v \circ h_v \circ \pi_v$. But since $y \in St^2(v) \setminus St(v)$ we see that $\pi_v(x) \neq \pi_v(y)$. Then since $h_v$ and $\iota_v$ are embeddings we have that $\beta_i(x) \neq \beta_i(y)$. Hence $\lambda$ is an embedding and therefore the proof of Theorem 1.3 is complete.

## 6. Equivalence of the Definitions of a Pseudo-Metric Polyhedron

6.1. **Assignment of the Quadratic Form.** Let $x \in \mathcal{X}$ be a point. Then there is a unique $k$-dimensional simplex $S_x =< v_0, v_1, ..., v_k > \in \mathcal{T}$ such that $x$ is interior to $S_x$. So we can consider a $k$-dimensional tangent space at $x$, denoted by $T_x$, whose dimension certainly depends on the triangulation $\mathcal{T}$. Under the simplicial isometric embeddings produced in sections 3, 4 and 5 we can consider $T_x$ as a $k$-dimensional affine subspace of $\mathbb{R}^p_q \cong \mathbb{R}^N$ where $N = p + q$. Consider the collection of vectors

$$\mathcal{B}_x = \{v_1 - v_0, v_2 - v_0, ..., v_k - v_0\}$$

where the difference makes sense since we are considering the complex as a subspace of $\mathbb{R}^N$. Clearly $\mathcal{B}_x$ is a basis for $T_x$. So to $S_x$ we associate the $k \times k$ symmetric matrix $G(S_x)$ defined by:

$$G(S_x)_{ij} = (< (v_i - v_0), (v_j - v_0) >)_{ij}$$

where the inner product is take in $\mathbb{R}^p_q$. At first glance it seems that this definition might depend on the embedding function. What we do now is (quickly) show that this is not true. The key is that:

[7]where we consider $e$ as the edge between $u$ and $w*$ in $\mathcal{S}_w$

[8]The purpose for gluing extra simplices onto $St(v)$ in the construction of $\mathcal{S}_v$ was so that we can extend this map simplicially over all of $\mathcal{X}$

$$g^2(e_{ij}) = < (v_i - v_j), (v_i - v_j) > = < ((v_i - v_0) - (v_j - v_0)), ((v_i - v_0) - (v_j - v_0)) >$$
$$= g^2(e_{0i}) - 2 < (v_i - v_0), (v_j - v_0) > + g^2(e_{0j}).$$

So

$$< (v_i - v_0), (v_j - v_0) > = \frac{1}{2}\left(g^2(e_{0i}) + g^2(e_{0j}) - g^2(e_{ij})\right)$$

where the edge notation is the same as always.

This shows that the matrix $G(S_x)$ depends only on the intrinsic metric $g$. Of course, $G(S_x)$ also depends on how we ordered the vertices of $S_x$. But changing the order of the vertices of $S_x$ just changes the coordinates of $\mathcal{B}_x$. Thus $G(S_x)$ is well-defined when considered as a symmetric bilinear form on $T_x$ (or equivalently on the tangent space to any point interior to $S_x$).

In this way we associate a symmetric bilinear form to every simplex of $\mathcal{T}$. This form allows us to assign an *energy* to any straignt line segment interior to any closed simplex. For if $S \in \mathcal{T}$ is a $k$-dimensional simplex and $a, b \in S$ with barycentric coordinates $(\alpha_i)_{i=0}^k$ and $(\beta_i)_{i=0}^k$ respectively then the energy of the straight line segment (in $S$) from $a$ to $b$ is

$$v^T G(S) v$$

where $v \in \mathbb{R}^k$ is defined as

$$v = (\alpha_i - \beta_i)_{i=1}^k$$

It is easy to see that the energy of a line segment is well-defined at the intersection of any simplices. It is also easy to see that the energy assigned to any edge $e_{ij}$ under this definition is $g^2(e_{ij})$. Thus the collection of pseudo-metrics on $(\mathcal{X}, \mathcal{T})$ is in one-to-one correspondence with assignments of a symmetric bilinear form to each simplex of $\mathcal{T}$ that agree (meaning they assign the same energy to any line segment) on the intersection of any two simplices.

### 6.2. **Euclidean and Minkowski Polyhedra.**

**Definition 6.1.** Let $(\mathcal{X}, \mathcal{T}, g)$ be a pseudo-metric polyhedron and let $G$ be the symmetric bilinear form defined as above with respect to $g$. $\mathcal{X}$ is a *Euclidean Polyhedron* if $G(S)$ is positive definite for all $S \in \mathcal{T}$. $\mathcal{X}$ is a *Minkowski Polyhedron* if $G(S)$ is non-degenerate for all $S \in \mathcal{T}$

A $k$-dimensional simplex $S \in \mathcal{T}$ admits a simplicial isometric embedding into Euclidean space of dimension $k$ if and only if $G(S)$ is positive definite for all $S \in \mathcal{T}$. For a proof see [2]. If a $k$-dimensional simplex admits a simplicial isometric embedding into $\mathbb{R}^p_q$ with $p + q = k$ then the signature of $G(S)$ will not contain any zeroes since the inner product on $\mathbb{R}^p_q$ is non-degenerate. This justifies the above definition.

Note that for a general metric polyhedron the quadratic form $G(S)$ can have zeroes in its signature. Theorems 1.1, 1.2 and 1.3 do not contradict the above statement since $p + q > n$ in these Theorems.

*Proof of Theorem 1.4.* This proof is another trick due to Greene [3]. Let $(\mathcal{X}, \mathcal{T}, g)$ be an $n$-dimensional metric polyhedron, where the triangulation $\mathcal{T}$ is locally finite. For any arbitrary simplex $S$ of $\mathcal{T}$, let $G$ denote the symmetric bilinear form determined by $g$ associated with $S$. Let $h \in \mathrm{Simp}(\mathcal{X}, \mathbb{E}^{2n+1})$ be an embedding and let $G(h)$ denote the symmetric bilinear form induced by the map $h$ on an arbitrary simplex $S$. If $\mathcal{X}$ is compact, then we can simply scale $h$ by a large constant so that $G(h) + G$ is positive definite on every simplex of $\mathcal{T}$. If $\mathcal{X}$ is not compact, then a simplicial map $h$ such that $h$ is an embedding and $G(h) + G$ is positive definite still exists. To see

this, just fix a vertex $v$ of $\mathcal{T}$ and continue scaling vertices of $\mathcal{T}$ which are "farther and farther away" from $v$ by "larger and larger" positive constants. Such a construction is not very difficult to put together. The resulting map may not be an embedding, but we can "wiggle" the vertices slightly so that $h$ is an embedding and we still have $G(h) + G$ is positive definite. For a proof, please see [7].

Now by Krat [6] there exists a map $f : \mathcal{X} \to \mathbb{E}^n$ which is simplicial on a subdivision $\mathcal{T}'$ of $\mathcal{T}$ and satisfies that $G(f) = G(h) + G$ with respect to every simplex of $\mathcal{T}'$. Then $G(f) - G(h) = G$, so the map $f \oplus h : \mathcal{X} \to \mathbb{R}^n_{2n+1}$ is an isometric embedding which is simplicial on $\mathcal{T}'$.

□

## References


1. A. V. Akopyan, *PL-analogue of Nash-Keiper theorem*, preliminary version (in Russian): http://www.moebiuscontest.ru/files/2007/akopyan.pdf www.moebiuscontest.ru
2. Rajendra Bhatia, *Positive Definite Matrices*, Princeton Series in Applied Mathematics (2007), pp. 2-3
3. Robert E. Greene, *Isometric Embeddings of Riemannian And Pseudo-Riemannian Manifolds*, American Mathematical Society (1970), pp. 36
4. M.L. Gromov and V.A. Rokhlin, *Embeddings And Immersions In Riemannian Geometry*, Russ. Math. Surv., Vol. 25, No. 1 (1970)
5. J. Hocking and G. Young, *Topology*, Dover (1988), 214–215.
6. S. Krat, *Approximation problems in length geometry*, Ph. D. Thesis, The Pennsylvania State University (2004)
7. B. Minemyer, *Isometric Embeddings of Euclidean Polyhedra*, in preparation, arXiv identifier 1211.0586
8. John Nash, *$C^1$ Isometric Imbeddings*, The Annals of Mathematics, Second Series, Vol. 60 (1954), pp. 383-396
9. John Nash, *The Imbedding Problem for Riemannian Manifolds*, The Annals of Mathematics, Second Series, Vol. 63, No. 1 (1956), pp. 20-63.
10. V. A. Zalgaller, *Isometric imbedding of polyhedra*, Dokl. Akad. Nauk SSSR (in Russian), Vol. 123 (1958), No. 4, pp. 599-601


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